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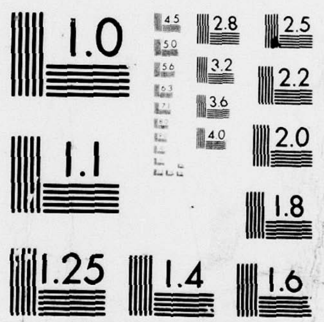

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NEARLY BALANCED INCOMPLETE BLOCK DESIGNS

Ching-Shui Cheng and Chien-Fu Wu

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NEARLY BALANCED INCOMPLETE BLOCK DESIGNS

Ching-Shui Cheng[1] and Chien-Fu Wu[2]

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ABSTRACT

A class of nearly balanced incomplete block designs is defined. This extends the concept of regular graph designs of John and Mitchell (1977) to the unequally replicated case. Some necessary conditions on the existence of such designs are derived. Methods of construction are given for some special cases. For five or six varieties, the "best" nearly balanced incomplete block designs and their efficiencies are tabulated.

AMS (MOS) Subject Classifications - Primary 62K10, 05B05

Key Words - A-efficiency, D-efficiency, nearly balanced incomplete block designs, regular graph designs

Work Unit Number 4 - Probability, Statistics, and Combinatorics

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SIGNIFICANCE AND EXPLANATION

To control the variations in experiments for comparing several treatments, one often uses the technique of block designs. Traditional work on the methods of construction has been concentrated on the equally replicated case, that is, the total number of experimental units is a multiple of the total number of treatments to be compared. In this case, the balanced incomplete block designs, when they exist, are most efficient. We consider extensions of this class of designs to the unequally replicated case. The idea is to relate some important features of the designs to graphs. Some methods of construction of the proposed "nearly balanced incomplete block designs" can be obtained via this connection. A list of the "best" nearly balanced incomplete block designs for small parameters is given.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NEARLY BALANCED INCOMPLETE BLOCK DESIGNS

Ching-Shui Cheng[1] and Chien-Fu Wu[2]

1. INTRODUCTION.

Traditional work on the construction of block designs has been concentrated on the equally replicated case, i.e., the total number of experimental units is a multiple of the total number of varieties. For a given number of varieties to be compared and a given block size, the assumption of equal replication imposes a severe constraint on the number of blocks which can be constructed. From a practical viewpoint, it is desirable to have a method to construct efficient block designs for the unequally replicated case. This paper constitutes an attempt to this type of problems.

The key idea is to choose designs which are as "balanced" as possible. For the equally replicated case, John and Mitchell (1976, 1977) have defined the regular graph designs to be a class of binary incomplete block designs, where every pair of varieties appeared in λ_1 or λ_2 blocks with $\lambda_2 = \lambda_1$ or $\lambda_1 + 1$. Although proposed on an intuitive basis, this class of designs turned out to be very efficient. See Cheng (1978b) for a study of the efficiency of regular graph designs. John and Mitchell have also conjectured that the "best" regular graph design, if exists, is also the "best" over the whole class of incomplete block designs. This conjecture was confirmed in Cheng (1978a) for some special cases. Inspired by the success of these works, we extend John and Mitchell's idea to the unequally replicated case and define a class of nearly balanced incomplete block designs. The definition is given in Section 2. The regular graph design introduced by John and Mitchell can be related to a graph with one common degree. Similarly, the nearly balanced incomplete block design can be related to a graph with at most two different degrees. As a consequence, we have

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derived some necessary conditions for the existence of such designs in Section 3. For block size two or $v - 1$, where v is the number of varieties, nearly balanced incomplete block designs exist for any number of blocks. Simple methods of construction are given in Section 5. For five or six varieties, a list of best nearly balanced incomplete block designs and their efficiency lower bounds is given in Table 1. For four varieties, the optimality of block designs with nearly balanced structure has been considered in Cheng (1978c).

2. DEFINITION AND CLASSIFICATION OF NEARLY BALANCED INCOMPLETE BLOCK DESIGNS.

When bk is a multiple of v, a balanced incomplete block design with v varieties and b blocks of size k is an incomplete block design with each variety appearing in each block at most once, each variety appearing in r blocks and any two varieties appearing in λ blocks, where $r = bk/v$ and $\lambda = r(k-1)/(v-1)$. Such designs, if exist, are optimal in a very strong sense (Kiefer, 1975). For arbitrary v,k,b, such designs may not exist. But, still, we want to get designs whose combinatorial properties are as close to those of the balanced incomplete block designs as possible. This motivates the following.

Definition. A nearly balanced incomplete block design (NBIBD) with v varieties and b blocks of size k (v does not necessarily divide bk) is an incomplete block design satisfying the following conditions:

(i) Each variety appears in each block at most once.

(ii) Let r_i be the number of replications of variety i. Then each $r_i = \bar{r}$ or $\bar{r} + 1$, where $\bar{r} = [bk/v]$ is the integral part of bk/v.

(iii) Let λ_{ij} be the number of times varieties i and j appear together. Then for each fixed i_0, $|\lambda_{i_0 j} - \lambda_{i_0 j'}| \leq 1$ for any $j \neq j'$, $j,j' \neq i_0$.

These designs can be further classified into two types. Let s be the number of varieties i with $r_i = \bar{r}$. Then

$$s = v - (bk - v\bar{r}) . \tag{2.1}$$

Note that $1 \leq s \leq v$ and when $s = v$, the design is a regular graph design. It is easily seen from the definition that if $r_{i_0} = \bar{r}$, then variety i_0 appears together with any other variety $\bar{\lambda}$ or $\bar{\lambda} + 1$ times, where $\bar{\lambda}$ is the integral part of $(k-1)\bar{r}/(v-1)$. In this case, the number of varieties each of which appears together with variety i_0 $\bar{\lambda}$ times is given by

$$n = v - 1 - \{(k-1)\bar{r} - \bar{\lambda}(v-1)\} . \tag{2.2}$$

As to the varieties i_0 with $r_{i_0} = \bar{r} + 1$, there are two possibilities according to $n \geq k - 1$ or $n < k - 1$. If $n \geq k - 1$, then for $r_{i_0} = \bar{r} + 1$, there are

$n - k + 1$ varieties each of which appears together with variety i_0 $\bar{\lambda}$ times and there are $v + k - n - 2$ $\lambda_{i_0 j}$'s equal to $\bar{\lambda} + 1$. On the other hand, if $n < k - 1$, then for $r_{i_0} = \bar{r} + 1$, there are $v - k + n$ varieties each of which appears together with variety i_0 $\bar{\lambda} + 1$ times and there are $k - 1 - n$ $\lambda_{i_0 j}$'s equal to $\bar{\lambda} + 2$. Thus if $n \geq k - 1$, then there are only two possible values of λ_{ij}, i.e., $\bar{\lambda}$ or $\bar{\lambda} + 1$. Such a nearly balanced incomplete block design is said to be of type I. For $n < k - 1$, λ_{ij} can be $\bar{\lambda}$, $\bar{\lambda} + 1$, or $\bar{\lambda} + 2$. In this case, we have a nearly balanced incomplete block design of type II.

3. RELATIONS TO GRAPHS AND NECESSARY CONDITIONS FOR EXISTENCE.

In this section, we relate nearly balanced incomplete block designs to graphs. The existence of a nearly balanced incomplete block design implies the existence of a graph with at most two distinct degrees which can be determined from the values of $v, b,$ and k. Using a result of Erdős and Gallai (1960) concerning the existence of graphs with prescribed degrees, we then derive a necessary condition for the existence of nearly balanced incomplete block designs.

For a binary incomplete block design with fixed block size k, $\{r_i\}_{i=1}^{v}$ and $\{\lambda_{ij}\}$, $i \neq j$, $1 \leq i, j \leq v$, the variance-covariance matrix for estimating the treatment effects is proportional to the generalized inverse of the so-called C matrix, whose diagonal elements are $(1 - k^{-1})r_i$, $i = 1,\ldots,v$ and off-diagonal elements $-k^{-1}\lambda_{ij}$, $1 \leq i, j \leq v$. The optimality criteria for block designs are usually defined in terms of this C matrix, as will be done in Section 4.

Let d be a nearly balanced incomplete block design of type I. Without loss of generality, we may assume that $r_1 = r_2 = \ldots = r_s = \bar{r}$ and $r_{s+1} = \ldots = r_v = \bar{r} + 1$. Let

$$A = \begin{bmatrix} \{(k-1)\bar{r} + \bar{\lambda}\}I_s - \bar{\lambda}J_{s,s} & -\bar{\lambda}J_{s,v-s} \\ -\bar{\lambda}J_{v-s,s} & \{(k-1)(\bar{r}+1) + \bar{\lambda}\}I_s - \bar{\lambda}J_{v-s,v-s} \end{bmatrix}, \quad (3.1)$$

where I_s is the $s \times s$ identity matrix and $J_{s,s}$ is the $s \times s$ matrix of ones. Then $A - kC$, where C is the matrix defined in the above paragraph, is a matrix in which all the diagonal elements are zero and the off-diagonals are zero or one. Each of the first s rows has $v - n - 1$ 1's and each of the last $v - s$ rows has $v - n + k - 2$ 1's. Thus $A - kC$ is the adjacency matrix of a graph with v vertices in which s vertices have degree $v - n - 1$ and $v - s$ vertices have degree $v - n + k - 2$.

Erdős and Gallai (1960) proved that a graph with prescribed degree sequence $d_1 \geq d_2 \geq \ldots \geq d_v$ exists if and only if

$$\text{(i)} \quad \sum_{i=1}^{v} d_i \text{ is even,} \qquad (3.2)$$

and (ii) $\sum_{i=1}^{\ell} d_i \leq \ell(\ell - 1) + \sum_{i=\ell+1}^{v} \min\{\ell, d_i\}$ for each integer ℓ with (3.3)

$1 \leq \ell \leq v - 1$

See, e.g., Harary (1969), p. 59.

Using (3.2) and (3.3), one can show that there exists a graph with v vertices in which s vertices have degree x and $v - s$ vertices have degree y with $x < y$ if and only if (i) $xs + (v - s)y$ is even and

$$\text{(ii)} \quad (v - s)y \leq (v - s)(v - s - 1) + sx \,. \tag{3.4}$$

In the present case, $x = v - n - 1$ and $y = v - n + k - 2$. From (2.1) and (2.2), $xs + (v - s)y = (v - n - 1)s + (v - s)(v - n + k - 2) = bk(k - 1) - \bar{\lambda}v(v - 1)$, which clearly is even. Therefore we conclude

<u>Proposition 1</u>. For any given positive integers b,v,k with $k \geq 2$ and $k < v$, let s and n be defined as in (2.1) and (2.2), respectively. If $n \geq k - 1$, and there exists a nearly balanced incomplete block design with v varieties and b blocks of size k, then $s(n - s + 1) \leq (v - s)(n - k + 1)$.

This provides a necessary condition for the existence of a nearly balanced incomplete block design of type I. For example, if $v = 5$, $k = 3$, $b = 3$, then $\bar{r} = 1$, $s = 1$, and $n = 2$. In this case, $n \geq k - 1$ but $s(n - s + 1) > (v - s)(n - k + 1)$. Therefore, no nearly balanced incomplete block design exists. However, if $v = 5$, $k = 3$, $b = 8$, then $\bar{r} = 4$, $s = 1$, $n = 4$ and $s(n - s + 1) \leq (v - s)(n - k + 1)$ holds. The associated graph has one vertex with degree four and four vertices with degree two and is unique. The corresponding design is therefore unique and is given in Table 1.

Next we consider a nearly balanced incomplete block design of type II. Again we assume that $r_1 = r_2 = \ldots = r_s = \bar{r}$ and $r_{s+1} = \ldots = r_v = \bar{r} + 1$. For any two different vertices i and j with $1 \leq i \leq s$ and $s + 1 \leq j \leq v$, we must have $\lambda_{ij} = \bar{\lambda}$ or $\bar{\lambda} + 1$. On the other hand, $\lambda_{ij} = \lambda_{ji} = \bar{\lambda} + 1$ or $\bar{\lambda} + 2$. Therefore λ_{ij} must be equal to $\bar{\lambda} + 1$. Thus if

$$B = \begin{bmatrix} \{(k-1)\bar{r} + \bar{\lambda} + 1\}I_s - (\bar{\lambda}+1)J_{s,s} & -(\bar{\lambda}+1)J_{s,v-s} \\ -(\bar{\lambda}+1)J_{v-s,s} & \{(k-1)(\bar{r}+1) + (\bar{\lambda}+2)\}I_{v-s} - (\bar{\lambda}+2)J_{v-s,v-s} \end{bmatrix} \quad (3.5)$$

then

$$kC - B = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \quad (3.6)$$

where N is $s \times s$, M is $(v-s) \times (v-s)$, and both M and N are zero-one matrices with all the diagonal elements equal to zero. Furthermore, each row of N has n 1's and each row of M has $v - k + n - s$ 1's. Therefore N is the adjacency matrix of a regular graph with s vertices and degree n, and M is the adjacency matrix of a regular graph with $v - s$ vertices and degree $v - k + n - s$. Such regular graphs exist if and only if $0 \leq n \leq s - 1$, $0 \leq v - k + n - s \leq v - s - 1$, and both ns and $(v-s)(v-k+n-s)$ are even.

In summary, we have

Proposition 2. For any given positive integers b,v,k with $k \geq 2$ and $k < v$, let s and n be defined as in (2.1) and (2.2), respectively. If $n < k - 1$, and there exists a nearly balanced incomplete block design with v varieties and b blocks of size k, then $n+1 \leq s \leq v-k+n$, and both ns and $(v-s)(v-k+n-s)$ are even

This provides a necessary condition for the existence of a nearly balanced incomplete block design of type II. For example, when $v = 6$, $k = 4$, $b = 5$, we have $\bar{r} = 3$, $s = 4$, $n = 1$. In this case $n < k - 1$; however, since $s > v - k + n$ a nearly balanced incomplete block design does not exist. Another example: for $v = 6$, $k = 3$, $b = 15$, $sn = 3$ is odd and hence the design does not exist.

By the above discussion, we see that a graph can always be constructed from a design. Unfortunately, this process need not be reversible. The existence of a graph does not imply that a design exists. Even if a design exists, it might be difficult to be constructed from the corresponding graph. One exception is $k = 2$. When block size is two, a design is equivalent to a graph since each block can be considered as an edge. Using Proposition 1, we can prove

Proposition 3. For any given positive integers b and v, a nearly balanced incomplete block design with block size two always exists and is of type I.

Proof. By (2.2), $n \geq 1 = k - 1$. Hence if the block size is two, then a nearly balanced incomplete block design must be of type I. By Proposition 1 and the equivalence of graphs and designs with $k = 2$, such a design exists if $s(n-s+1) \leq (v-s)(n-1)$. When $s > n$, $s(n-s+1) \leq 0 \leq (v-s)(n-1)$; while for $s \leq n-1$, $s(n-s+1) \leq (n-1)(v-s)$ since $n-s+1 \leq v-s$. Therefore it remains to show that the inequality holds for $s = n$, that is to show that $n \leq (v-n)(n-1)$. In this case, from the paragraph right before Proposition 1, $vn - (v-n)(k-1) = vn - (v-n)$ is even. It follows that both v and n are even. Together with $1 \leq n \leq v-1$, this implies $2 \leq n \leq v-2$. The inequality $n \leq (v-n)(n-1)$ clearly follows. Q.E.D.

Proposition 3 guarantees the existence of a nearly balanced incomplete block design for $k = 2$ and any values of b and v. Two methods of construction are described in Section 5.

4. EFFICIENCY BOUNDS.

For an incomplete block design d, let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{v-1} \geq \mu_v = 0$ be the eigenvalues of the C matrix of d. Define $\Phi_A(d) = \sum_{i=1}^{v-1} \mu_i^{-1}$ and $\Phi_D(d) = \prod_{i=1}^{v-1} \mu_i^{-1}$. A design is A- (or D-) optimal if it minimizes the Φ_A (or Φ_D) values among all the possible designs with the same parameters (v,k,b). The A-efficiency and D-efficiency of a design d is defined to be $e_A(d) = \Phi_A(\text{A-optimal design})/\Phi_A(d)$ and $e_D(d) = \{\Phi_D(\text{D-optimal design})/\Phi_D(d)\}^{\frac{1}{v-1}}$. One problem with these definitions is that optimal designs are only known for some special cases. Instead, we will give simple lower bounds of e_A and e_D as some conservative measures of the efficiencies of d. For any design d with parameters (v,k,b), Kiefer (1958, 1975) has shown that $\Phi_A(d) \geq (v-1)^2\delta^{-1}$ and $\Phi_D(d) \geq \{\delta^{-1}(v-1)\}^{v-1}$, where $\delta = b(k-1)$. These two lower bounds are the Φ_A- and Φ_D-values of a balanced incomplete block design, if exists, with parameters (v,k,b). Two efficiency lower bounds of e_A and e_D are defined as

$$e'_A(d) = (v-1)^2/\delta\Phi_A(d) \tag{4.1}$$

and

$$e'_D(d) = \delta^{-1}(v-1)/\{\Phi_D(d)\}^{\frac{1}{v-1}} \tag{4.2}$$

These will be used to measure the efficiencies of the proposed designs of the paper in Section 6.

5. CONSTRUCTION.

A block design with v varieties and b blocks of size two is fully equivalent to a graph with v vertices and b edges. Note that each edge connects two vertices in the same way as each block contains two varieties. Therefore, a nearly balanced incomplete block design with v varieties and b blocks of size two, $b \le \frac{1}{2} v(v - 1)$, can be viewed as a graph with v vertices and b edges such that, for any pair of vertices, there is at most one edge connecting them, that s vertices have degree $\bar{r}$ and $v - s$ vertices have degree $\bar{r} + 1$, where $\bar{r} = [\frac{2b}{v}]$ and $s = v\bar{r} + v - 2b$. The existence of such graphs has been proved in Section 3.

The following scheme gives a method of construction of graphs with a prescribed degree sequence. From a result due to Hakimi and Havel (Harary, 1969, p. 58), a graph d with v vertices and degrees $d_1 \ge d_2 \ge \ldots \ge d_v$ exists if a graph d' with $v - 1$ vertices and degrees $(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_v)$ exists. Graph d can be obtained from graph d' by introducing one more vertex to d' and connecting it to the d_1 vertices of d' with degrees $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1$. For example, for $v = 7$, $k = 2$, $b = 8$, $\bar{r} = 2$ and $s = 5$. From the above iterative scheme, the corresponding degree sequences are $(3,3,2,2,2,2,2)$, $(2,2,2,2,1,1)$, $(2,1,1,1,1)$ and $(1,1,0,0)$. The building-up process is illustrated in Figure 1.

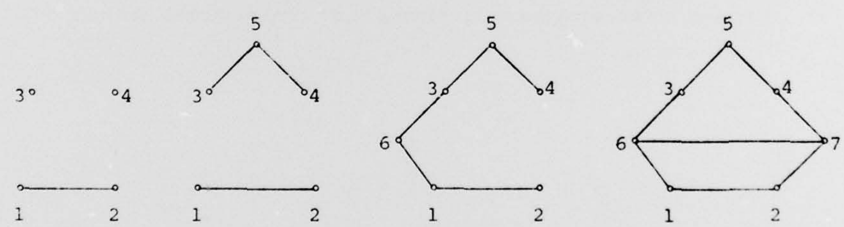

FIGURE 1

Another method of construction, which is non-iterative in nature, is outlined below. A hamiltonian cycle is a collection of edges of the form (i_1,i_2), $(i_2,i_3), \ldots,$ (i_{v-1},i_v), (i_v,i_1), where $(i_1,\ldots,i_v)$ is a permutation of $(1,\ldots,v)$. For v odd, a complete graph of v vertices is the union of $\frac{v-1}{2}$ disjoint hamiltonian cycles $\{S_i\}$. See Theorem 9.6 of Harary (1969). Suppose $(q-1)v < b \leq qv$ for some positive integer $q \leq \frac{v-1}{2}$. Then a nearly balanced incomplete block design with v varieties and b blocks of size two can be constructed as the sum of $S_1, \ldots, S_{q-1}$ and some $b-(q-1)v$ edges from S_q. For example, (16253471), (13645721) and (14237651) are three disjoint hamiltonian cycles for $v = 7$. When $b = 11$, one such design is $\{(16), (62), (25), (53), (34), (47), (71), (13), (64), (57), (21)\}$. For v even, a construction method can be obtained by modifying the above method slightly. For details, see Theorem 9.7 of Harary (1969).

We should remark that none of these methods of construction gives unique designs. For $c\frac{v(v-1)}{2} \leq b < (c+1)\frac{v(v-1)}{2}$, where c is an integer, the construction of a nearly balanced incomplete block design with v varieties and b blocks of size two can be reduced to the case $b < \frac{1}{2}v(v-1)$ by adding c copies of the balanced incomplete block design with $\frac{1}{2}v(v-1)$ blocks.

For $k = v - 1$, the construction of nearly balanced incomplete block designs is very simple. Starting with the balanced incomplete block design d^* with v blocks, a nearly balanced incomplete block design with b blocks, $2 \leq b < v$, can be obtained as any b blocks of d^*. For $v < b \leq 2v$, one simply adds any $b - v$ blocks of d^* to d^*.

For $3 \leq k \leq v - 2$, a nearly balanced incomplete block design may not exist. The problem of constructing such designs is still very much open. But when b is close to b^* for which a balanced incomplete block design d^* with b^* blocks exists, a nearly balanced incomplete block design with b blocks may be obtained by adding or removing some blocks to or from d^*.

6. EXAMPLES.

A list of nearly balanced incomplete block designs for $v = 5,6$ is given in Table 1. The case $k = v - 1$ is not considered here since it has been treated in Section 5. For the equally replicated case, i.e. bk is divisible by v, Mitchell and John (1976) have listed all the best regular graph designs. The readers should consult their paper for the description of designs and their efficiencies, whose definition is different from ours.

In Table 1, a balanced incomplete block design with parameters (v,k,b) is denoted by BIBD (v,k,b). The best regular graph design which is not BIBD is denoted by RGD. The meaning for NBIBD (v,k,b) is also evident. When there exist several designs for the same parameters, only the one with the highest e'_A and e'_D values (see formulae (4.1), (4.2)) is listed. For example, for $v = 6$, $k = 2$, $b = 7$, the corresponding graph as described in Section 5 has four vertices with degree two and two vertices with degree three. There are four non-isomorphic graphs with this property, Harary (1969). The four associated designs are $d_1 = (12)(23)(34)(45)(56)(61)(36)$, $d_2 = (12)(23)(34)(45)(56)(61)(15)$, $d_3 = (12)(23)(34)(45)(51)(16)(36)$ and $d_4 = (12)(23)(31)(16)(45)(56)(64)$. Their A- and D-efficiency lower bounds e'_A and e'_D are, respectively, (0.754, 0.878), (0.731, 0.866), (0.793, 0.889), (0.510, 0.793). Therefore, only d_3 is listed in Table 1. If there exist a BIBD (v,k,b_1) and an NBIBD (v,k,b_2), then an NBIBD $(v,k,b_1 + b_2)$ can be obtained by combining these two designs. This is useful for constructing nearly balanced incomplete block designs with large b. For example, an NBIBD $(5,2,14)$ can be obtained by combining the BIBD $(5,2,10)$ and an NBIBD $(5,2,4)$.

The high efficiencies of the "best" nearly balanced incomplete block designs are evident from Table 1. In fact, we can obtain a sharp lower bound of the A- and D-efficiences of the whole class of nearly balanced incomplete block designs for a given parameter set. These efficiency lower bounds turn out to be very high as was experienced in Cheng (1978b). For the sake of simplicity, these are omitted in the paper.

TABLE 1. SOME NEARLY BALANCED INCOMPLETE BLOCK DESIGNS AND THEIR A-, D-EFFICIENCY LOWER BOUNDS: UNEQUALLY REPLICATED CASE

RGD: regular graph design exists.

No NBIBD: no nearly balanced incomplete block design exists.

Number of Blocks	Design d	$e'_A(d)$	$e'_D(d)$
	(i) v = 5, k = 2		
4	(12)(23)(34)(45)	0.500	0.747
5	RGD		
6	(12)(23)(34)(41)(25)(45)	0.869	0.927
7	(12)(23)(34)(45)(51)(13)(24)	0.890	0.945
8	BIBD (5,2,10) minus (23)(45)	0.937	0.968
9	Any nine blocks of the BIBD (5,2,10)	0.952	0.977
10	BIBD (5,2,10)	1.0	1.0
11	BIBD (5,2,10) plus (12)	0.979	0.988
12	BIBD (5,2,10) plus (12)(34)	0.972	0.986
13	BIBD (5,2,10) plus (12)(34)(15)	0.969	0.984
	(ii) v = 5, k = 3		
2	(123)(451)	0.714	0.863
3,4,6,7	No NBIBD		
5	RGD		
8	BIBD (5,3,10) minus (123)(451)	0.985	0.992
9	BIBD (5,3,10) minus (123)	0.987	0.993
10	BIBD (5,3,10)	1.0	1.0
11	BIBD (5,3,10) plus (123)	0.991	0.995

TABLE 1 (CONT.) SOME NEARLY BALANCED INCOMPLETE BLOCK DESIGNS AND THEIR A-, D-EFFICIENCY LOWER BOUNDS: UNEQUALLY REPLICATED CASE

Number of Blocks	Design d	$e'_A(d)$	$e'_D(d)$
	(iii) v = 6, k = 2		
5	(12)(23)(34)(45)(56)	0.428	0.715
6,9,12,18	RGD		
7	(12)(23)(34)(45)(51)(16)(36)	0.793	0.889
8	(12)(23)(34)(45)(56)(61)(14)(25)	0.833	0.915
10	(12)(23)(34)(45)(56)(61) (14)(15)(25)(36)	0.914	0.954
11	(12)(23)(34)(45)(56)(61) (15)(24)(35)(36)(46)	0.921	0.960
13	BIBD (6,2,15) minus (12)(34)	0.961	0.981
14	BIBD (6,2,15) minus (12)	0.974	0.988
15	BIBD (6,2,15)	1.0	1.0
16	BIBD (6,2,15) plus (12)	0.986	0.993
17	BIBD (6,2,15) plus (12)(34)	0.980	0.990
19	BIBD (6,2,15) plus (15)(25)(36)(46)	0.975	0.987
	(iv) v = 6, k = 3		
3	(156)(246)(345)	0.769	0.882
4,6,8,12	RGD		
5	No NBIBD		
7	(126)(134)(145)(235)(246)(356)(456)	0.963	0.981
9	Any nine blocks of the BIBD (6,3,10)	0.980	0.990
10	BIBD (6,3,10)	1.0	1.0
11	BIBD (6,3,10) plus (123)	0.988	0.994

TABLE 1 (CONT.) SOME NEARLY BALANCED INCOMPLETE BLOCK DESIGNS AND THEIR A-, D-EFFICIENCY LOWER BOUNDS: UNEQUALLY REPLICATED CASE

Number of Blocks	Design d	$e'_A(d)$	$e'_D(d)$
	(v) v = 6, k = 4		
2	(1234)(3456)	0.806	0.903
3,6,9,12	RGD		
4,5,7,8,10,11	No NBIBD		
13	BIBD (6,4,15) minus (1234)(3456)	0.995	0.997
14	BIBD (6,4,15) minus (1234)	0.996	0.998
15	BIBD (6,4,15)	1.0	1.0
16	BIBD (6,4,15) plus (1234)	0.997	0.998

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A class of nearly balanced incomplete block designs is defined. This extends the concept of regular graph designs of John and Mitchell (1977) to the unequally replicated case. Some necessary conditions on the existence of such designs are derived. Methods of construction are given for some special cases. For five or six varieties, the "best" nearly balanced incomplete block designs and their efficiencies are tabulated.

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